

Percolation for the stable marriage of Poisson and Lebesgue

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Received 11 January 2006; received in revised form 25 July 2006; accepted 5 September 2006

Available online 25 September 2006

Abstract

Let Ξ be the set of points (we call the elements of Ξ centers) of a Poisson process in \mathbb{R}^d , $d \geq 2$, with unit intensity. Consider the allocation of \mathbb{R}^d to Ξ which is stable in the sense of the Gale–Shapley marriage problem and in which each center claims a region of volume $\alpha \leq 1$. We prove that there is no percolation in the set of claimed sites if α is small enough, and that, for high dimensions, there is percolation in the set of claimed sites if $\alpha < 1$ is large enough.

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Keywords: Multiscale percolation; Phase transition; Critical appetite

1. Introduction and results

The following model was considered in [1,2]. Whenever possible, we keep the same notation here. The elements of \mathbb{R}^d , $d \geq 2$, are called *sites*. We write $|\cdot|$ for the Euclidean norm and \mathcal{L} for the Lebesgue measure in \mathbb{R}^d . Let Ξ be the set of points of Poisson point process Π in \mathbb{R}^d with intensity λ (usually we will assume that $\lambda = 1$). The elements of Ξ are called *centers*. Let $\alpha \in [0, +\infty]$ be a parameter called the *appetite*. An *allocation* of \mathbb{R}^d to Ξ with appetite α is a measurable function $\psi : \mathbb{R}^d \rightarrow \Xi \cup \{\infty, \Delta\}$, such that $\mathcal{L}[\psi^{-1}(\Delta)] = 0$, and $\mathcal{L}[\psi^{-1}(\xi)] \leq \alpha$

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for all $\xi \in \Xi$. The set $\psi^{-1}(\xi)$ is called the *territory* of the center ξ . We say that ξ is *sated* if $\mathcal{L}[\psi^{-1}(\xi)] = \alpha$, and *unsated* otherwise. We say that a site x is *claimed* if $\psi(x) \in \Xi$, and *unclaimed* if $\psi(x) = \infty$. Here $\psi(x) = \infty$ means that the site x is unable to find any center willing to accept it, and $\psi(x) = \Delta$ means that x is unable to decide between two or more different centers (intuitively, this means that x is exactly on the frontier between the territories of different centers from Ξ). An \mathcal{L} -null set of sites with $\psi(x) = \Delta$ is allowed for technical reasons.

Stability of an allocation is defined in the following way. Let ξ be a center and let x be a site with $\psi(x) \notin \{\xi, \Delta\}$. We say that x *desires* ξ if

$$|x - \xi| < |x - \psi(x)| \quad \text{or } x \text{ is unclaimed.}$$

We say that ξ *covets* x , if

$$|x - \xi| < |x' - \xi| \quad \text{for some } x' \in \psi^{-1}(\xi) \text{ or } \xi \text{ is unsated.}$$

A site–center pair (x, ξ) is *unstable* for the allocation ψ if x desires ξ and ξ covets x . An allocation is *stable* if there are no unstable pairs.

The above definition of stable allocation is not constructive. A more constructive version can be found in Section 2 of [1]. Informally, the explicit construction of the stable allocation can be described as follows. For each center, we start growing a ball centered in it. All the balls grow simultaneously, at the same linear speed. Each center gets the sites captured by its ball, unless it is sated or the site was already captured by some other center. Remembering that one picture is worth a thousand words, we refer the reader to Fig. 1. Also, it is worth noting that the territory of a particular center is not necessarily connected (one can imagine the following situation: a center is surrounded by several other centers, so the territory it gets near itself is not enough, and so it has to wait until the neighbouring centers are sated to look for more territory outside).

In [1], among other results, there was proved the existence of stable allocation for any set of centers and any $\alpha \in [0, +\infty]$ and \mathbf{P} -a.s. \mathcal{L} -uniqueness of the stable allocation in both of the following cases:

- (i) Ξ is given by a set of points of an ergodic point process in \mathbb{R}^d or
- (ii) Ξ is finite.

Also, it was proved that

- if $\lambda\alpha < 1$ (subcritical regime) then a.s. all centers are sated but there is an infinite volume of unclaimed sites;
- if $\lambda\alpha = 1$ (critical regime) then a.s. all centers are sated and \mathcal{L} -a.a. sites are claimed;
- if $\lambda\alpha > 1$ (supercritical regime) then a.s. not all centers are sated but \mathcal{L} -a.a. sites are claimed.

Denote by \mathcal{C} the closure of $\psi^{-1}(\Xi)$. The set \mathcal{C} is the main object of study in this paper; it will be referred to as the *set of claimed sites* (even though it may contain some $x \in \mathbb{R}^d$ with $\psi(x) = \Delta$).

As shown in [1], this model has nice monotonicity properties, both in α and in Ξ (see Propositions 21 and 22 of [1]). In this paper, we only need some particular cases of what was proven there, namely,

- (i) if the sets \mathcal{C}_1 and \mathcal{C}_2 are constructed using the same set of centers Ξ and different appetites α_1 and α_2 respectively, and $\alpha_1 < \alpha_2$, then $\mathcal{C}_1 \subset \mathcal{C}_2$;
- (ii) if the sets \mathcal{C}_1 and \mathcal{C}_2 are constructed using the same appetite α and different sets of centers Ξ_1 and Ξ_2 respectively, and $\Xi_1 \subset \Xi_2$, then $\mathcal{C}_1 \subset \mathcal{C}_2$.

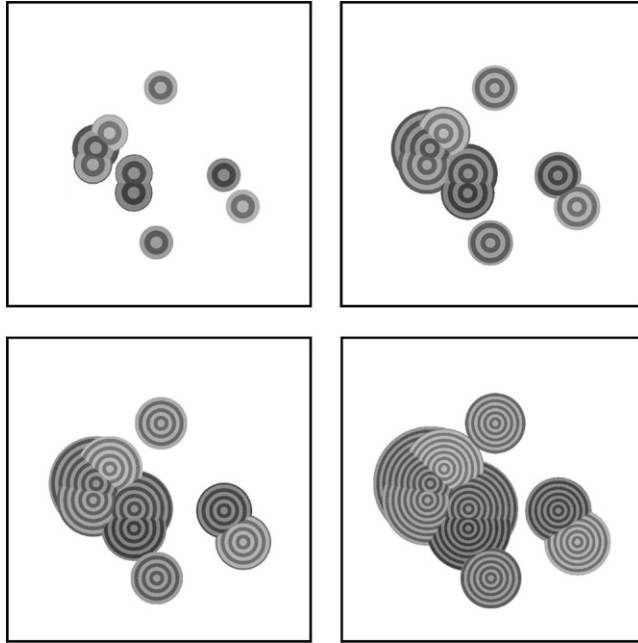


Fig. 1. Stable allocations for a *finite* configuration of centers Ξ , and with appetites $\alpha = 0.25, 0.45, 0.6, 0.8$.

In this paper we partially solve an open problem suggested in [1] concerning the percolation of the claimed sites.

Definition 1.1. We say that there is a percolation by claimed sites, if there exists an unbounded connected subset of \mathcal{C} .

Due to the monotonicity properties of the model, it is natural to define the percolation threshold $\alpha_p(d)$ in the following way:

$$\alpha_p(d) = \sup\{\alpha : \mathbf{P}[0 \text{ belongs to an unbounded connected subset of } \mathcal{C} \text{ in the } d\text{-dimensional model with appetite } \alpha] = 0\}.$$

On Fig. 2 one can see two configurations (inside a box 20×20 , with $\lambda = 1$) in (presumably) non-percolating and percolating phases.

Theorem 1.1. (i) For any dimension $d \geq 2$ we have that $\alpha_p(d) > 0$, that is, if the appetite α is small enough, then a.s. there is no percolation by claimed sites.
(ii) Also, if α is small enough and $d \geq 2$, then there exists percolation by unclaimed sites (i.e., a.s. there is an unbounded connected component in $\mathbb{R}^d \setminus \mathcal{C}$).

Since in the model with $\alpha = 1$ almost all the sites are claimed, that is, $\mathcal{C} = \mathbb{R}^d$, it is clear that $\alpha_p(d) \leq 1$. The next result implies that if the dimension is sufficiently high, then $\alpha_p(d) < 1$ (and even that $\alpha_p(d) \lesssim 2^{-d}$, as $d \rightarrow \infty$).

Theorem 1.2. We have

$$\limsup_{d \rightarrow \infty} \alpha_p(d) 2^d \leq 1. \quad (1.1)$$

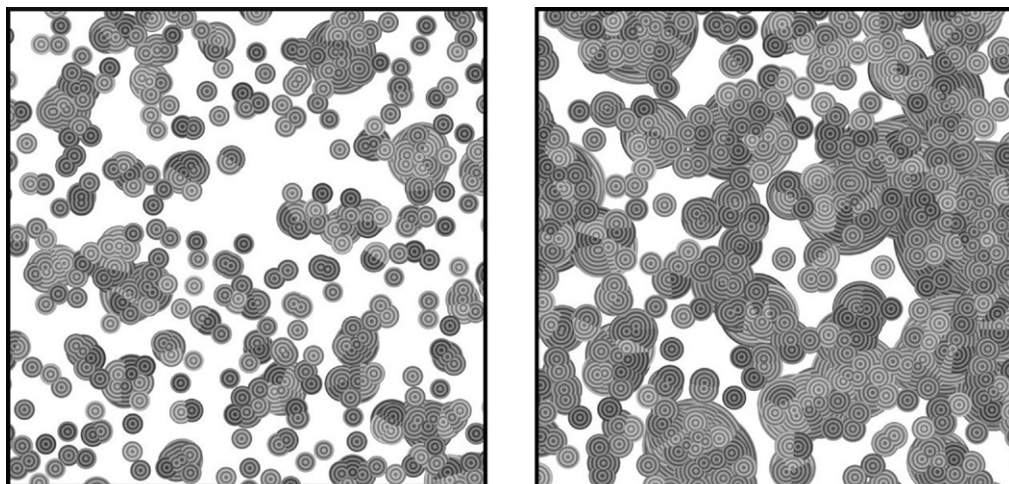


Fig. 2. In the left image one can see a realization of the model with $\alpha = 0.6$ (which seems to correspond to the non-percolating phase); in the right image, $\alpha = 0.8$ was used (which seems to correspond to the percolating phase).

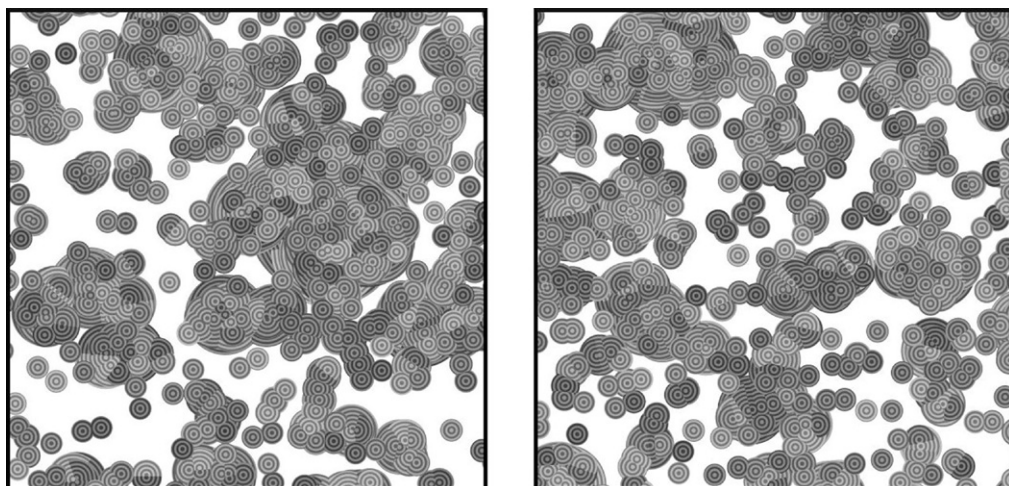


Fig. 3. Near the percolation threshold (two realizations with $\alpha = 0.7$): in the left image, crossings from left to right and from top to bottom do not exist; in the right image, there are crossings from left to right and from top to bottom.

Simulations suggest that $\alpha_p(2)$ is around 0.7 (see Fig. 3). Note, however, that proving that $\alpha_p(2) < 1$ (as well as $\alpha_p(d) < 1$ for small d) is still an open problem.

2. Proofs

Since the proof of Theorem 1.2 is much simpler, let us begin with

Proof of Theorem 1.2. Note that if we rescale the space by factor b (that is, apply a homothetic map $x \mapsto bx$), then we obtain the model with the intensity of the Poisson process being λ/b^d and the appetite αb^d . The geometric properties of the allocation do not change under this

transformation, and the product of intensity and appetite does not change either. In particular, this shows that the percolation properties of the model only depend on the product $\lambda\alpha$.

Let π_d be the volume of the unit ball in \mathbb{R}^d . Since the volume of the ball of radius $(\alpha/\pi_d)^{1/d}$ is α , any site which is at most $(\alpha/\pi_d)^{1/d}$ far away from some center will belong to \mathcal{C} . Indeed, the centers want territory of volume α as close as possible, so, for any center, any site x in the ball of volume α centered there will be claimed, either by this center, or by another one (or it may happen that $\psi(x) = \Delta$ so that x is disputed by two or more centers, but in this case $x \in \mathcal{C}$ anyway). So, the set of claimed sites \mathcal{C} dominates the Poisson Boolean model with rate $\lambda = 1$ and radius $(\alpha/\pi_d)^{1/d}$. By the above rescaling argument, that model is equivalent to the Poisson Boolean model with $\lambda = \alpha\pi_d^{-1}2^d$ and radius $1/2$. Let $\lambda_{cr}(d)$ be the critical rate for the percolation in the Poisson Boolean model with radius $1/2$. Now it is straightforward to obtain that [Theorem 1.2](#) is a consequence of the following result of [4]:

$$\lim_{d \rightarrow \infty} \pi_d \lambda_{cr}(d) = 1. \quad \square$$

To prove [Theorem 1.1](#), we need some preparations.

Let us from now on fix $\alpha = 1$ and vary λ , instead of fixing $\lambda = 1$ and varying α (the rescaling argument at the beginning of the proof of [Theorem 1.2](#) allows us to do this).

The idea of the proof of the part (i) of [Theorem 1.1](#) can be described as follows:

1. We define a “discrete” (i.e., made of cubes of size 1) dependent percolation model, and prove ([Lemma 2.1](#)) that it dominates the original model, so that it is enough to prove the absence of percolation in this discrete model.
2. The important properties of the discrete model are provided by (2.8) and [Lemma 2.2](#).
3. Then, in [Definition 2.1](#), we define the notion of a *passable* level- m cube (a level- m cube is a cube of size m ; see (2.1) below), and we show ([Lemma 2.3](#)) that the cubes are passable or not independently when they are far enough from each other.
4. Using that independence, we prove ([Lemmas 2.4](#) and [2.5](#)) that the probability that a bigger cube is passable can be bounded from above in terms of the probability that a smaller cube is passable.
5. This allows us to prove that, for small enough λ , the probability that a cube is passable tends to 0 as the size of the cube goes to infinity. With a little more work, this implies the absence of percolation.

For $m \geq 1$ and $i = (i^{(1)}, \dots, i^{(d)}) \in \mathbb{Z}^d$, define the *level- m cube* K_i^m associated with i by

$$K_i^m = \left\{ x = (x^{(1)}, \dots, x^{(d)}) \in \mathbb{R}^d : -\frac{m}{2} \leq x^{(l)} - mi^{(l)} \leq \frac{m}{2} \right\}. \quad (2.1)$$

Note that, in the above definition, the quantity m is not necessarily integer (although it is convenient to think about it as such). Note also that the union of all level- m cubes is \mathbb{R}^d , and the intersection of any two distinct level- m cubes is either empty, or has zero Lebesgue measure. We say that two cubes are connected if they have at least one point in common. Denote by $\zeta^{(i)}$ the number of centers in K_i^1 , i.e., the cardinality of the set $K_i^1 \cap \Xi$. At this point we need to introduce more notation. First, we define the distance between two sets $A, B \subset \mathbb{R}^d$ in a usual way:

$$\rho(A, B) = \inf_{x \in A, y \in B} |x - y|.$$

Then, for any $r \geq 0$, we define a discrete ball $B_i(r)$ by

$$B_i(r) = \bigcup_{j \in \mathcal{J}} K_j^1, \quad \text{if } r > 0$$

where

$$\mathcal{J} = \{j \in \mathbb{Z}^d : \rho(K_i^1, K_j^1) \leq r\},$$

and $B_i(0) := \emptyset$. We use the notation $\lceil x \rceil$ for the smallest integer greater than or equal to x and $\lfloor x \rfloor$ for the integer part of x . For each $i \in \mathbb{Z}^d$, define also the random variable (note that $\mathbb{Z}^d \cap K_j^1 = j$)

$$R_i = \inf \left\{ r > 0 : \sum_{j \in \mathbb{Z}^d \cap B_i(\beta_d r)} \zeta^{(j)} \leq \pi_d r^d \right\}, \quad \text{if } \zeta^{(i)} > 0 \quad (2.2)$$

(here we use the convention $\inf \emptyset = +\infty$), where, as before, π_d is the volume of the ball with radius 1 in \mathbb{R}^d ,

$$\beta_d = \lceil 3 + 2\sqrt{d}\pi_d^{1/d} \rceil, \quad (2.3)$$

and $R_i := 0$, if $\zeta^{(i)} = 0$.

We have the following.

Lemma 2.1. *The territories from all centers in K_i^1 are contained in $B_i(R_i)$.*

Proof of Lemma 2.1. Let us first show that

$$\rho(B_i(R_i), \mathbb{R}^d \setminus B_i(\beta_d R_i)) > R_i. \quad (2.4)$$

Indeed, since $\rho(K_i^1, B_i(R_i)) \leq R_i$ and the level-1 cubes have side 1 and thus diameter \sqrt{d} , we have

$$\max_{x \in K_i^1, y \in B_i(R_i)} |x - y| \leq R_i + 2\sqrt{d}. \quad (2.5)$$

For $\rho(B_i(R_i), \mathbb{R}^d \setminus B_i(\beta_d R_i))$, using (2.5) and the fact that $\rho(K_i^1, B_i(\beta_d R_i)) \leq \beta_d R_i$, we obtain

$$\rho(B_i(R_i), \mathbb{R}^d \setminus B_i(\beta_d R_i)) \geq \beta_d R_i - (R_i + 2\sqrt{d}). \quad (2.6)$$

Finally, note that if $R_i > 0$, then there is at least one center in $B_i(\beta_d R_i)$, and thus $\pi_d R_i^d \geq 1$, so $R_i \geq \pi_d^{-1/d}$. So, from (2.5) and (2.6), we get that if $\beta_d > 2 + 2\sqrt{d}\pi_d^{1/d}$ (by (2.3), this is indeed the case), then (2.4) holds.

Now, suppose that there exist $\xi \in K_i^1$ and $x \in \mathbb{R}^d$ such that $\psi(x) = \xi$ and $|x - \xi| > R_i$. One can choose a small enough ε such that $|x - \xi| > R_i + \varepsilon$ and any site z with $\rho(z, K_i^1) \leq R_i + \varepsilon$ belongs to $B_i(R_i)$ (this is possible since $B_i(R_i)$ is a compact set, and for any z' from the boundary of $B_i(R_i)$ it holds that $\rho(z', K_i^1) > R_i$; otherwise the next level-1 cube would be included in $B_i(R_i)$ too). There exists y such that $|y - \xi| \leq R_i + \varepsilon$ (and so $y \in B_i(R_i)$) and

- either $\psi(y) = \xi'$ for some $\xi' \in \mathbb{R}^d \setminus B_i(\beta_d R_i)$,
- or y is unclaimed.

This is because, by (2.2), the number of centers in $B_i(\beta_d R_i)$ is at most $\pi_d R_i^d$, and each one of them wants to claim a territory of volume 1, but $\mathcal{L}(\{z \in \mathbb{R}^d : |z - \xi| \leq R_i + \varepsilon\}) > \pi_d R_i^d$. Now, let us show that (y, ξ) is an unstable pair. Indeed,

- y desires ξ , because, by (2.4), we have $|y - \xi| < |y - \psi(y)|$, and
- ξ covets y , because $|y - \xi| < |x - \xi|$.

Thus, the centers from K_i^1 will be sated with territory inside $B_i(R_i)$ and Lemma 2.1 is proved. \square

Lemma 2.1 allows us to majorize the original model by the following (dependent) percolation model: given the set Ξ of points of Poisson process, for every K_i^1 we paint all the level-1 cubes in $B_i(R_i)$ and denote by \mathfrak{C} the set of painted sites. That is, we define

$$\mathfrak{C} = \bigcup_{i \in \mathbb{Z}^d} B_i(R_i).$$

At this point it is important to observe that, by Lemma 2.1, it holds that $\mathcal{C} \subset \mathfrak{C}$. Thus, to prove the first part of Theorem 1.1, it is sufficient to prove the absence of the infinite cluster in \mathfrak{C} for small λ .

Let us recall Chernoff's bound for Poisson random variable Z with parameter λ :

$$\mathbf{P}[Z > a] \leq e^{-\lambda g(\lambda/a)}, \quad (2.7)$$

where $g(x) = [x - 1 - \log x]/x$ (note that $g(x) \rightarrow +\infty$, as $x \rightarrow 0$). Since $\mathcal{L}(B_i(\beta_d a)) \leq (2\beta_d a + 3)^d$, by (2.2), we have

$$\begin{aligned} \mathbf{P}[R_i > a] &\leq \mathbf{P}\left[\sum_{j \in B_i(\beta_d a)} \zeta^{(j)} > \pi_d a^d\right] \\ &\leq \exp\left\{-\lambda(2\beta_d a + 3)^d g\left(\frac{\lambda(2\beta_d a + 3)^d}{\pi_d a^d}\right)\right\} \\ &\leq e^{-c(\lambda)a^d} \end{aligned} \quad (2.8)$$

where $c(\lambda) \rightarrow +\infty$, as $\lambda \rightarrow 0$ (a similar argument can be found in the proof of Proposition 11 from [2]).

The following simple fact is important for the proof of Theorem 1.1:

Lemma 2.2. *To determine whether the event $\{R_i \leq a\}$ occurs, we only have to look at the configuration of the centers inside $B_i(\beta_d a)$.*

Proof of Lemma 2.2. This follows immediately from the definition of R_i (see (2.2)). \square

Consider a bounded set $W \subset \mathbb{R}^d$ and let $\Xi_W = \Xi \cap W$ (since W is bounded, Ξ_W is a finite set a.s.). As noted above, there exists an a.s. unique stable allocation corresponding to the set of centers Ξ_W . We can then construct the set of painted sites $\mathfrak{C}|_W$ corresponding to this stable allocation analogously to the construction of \mathfrak{C} . Namely, first, we define the random variables $\zeta_W^{(i)}$ as the cardinality of the set $\Xi \cap K_i^1 \cap W$. Then, we define R_i^W analogously to (2.2) (only changing $\zeta^{(\cdot)}$ to $\zeta_W^{(\cdot)}$), and then we let $\mathfrak{C}|_W = \bigcup_{i \in \mathbb{Z}^d} B_i(R_i^W)$. From (2.2) it is straightforward to obtain that $\mathfrak{C}|_W \subset \mathfrak{C}$ for any $W \subset \mathbb{R}^d$.

Let K_j^m be a level- m cube and define

$$\mathcal{A}(K_j^m) = \bigcup_{i: K_i^m \cap K_j^m \neq \emptyset} K_i^m$$

(so, $\mathcal{A}(K_j^m)$ is the union of K_j^m with the $3^d - 1$ neighbouring level- m cubes). We use here some ideas typical for multiscale (fractal) percolation models; see e.g. [3]. First, we define the notion of *passable* cubes.

Definition 2.1. A level- m cube K_j^m is *passable* if

- (i) the set K_j^m intersects a connected component with diameter at least $m/2$ of $\mathfrak{C}|_{\mathcal{A}(K_j^m)}$, and
- (ii) for any $i \in \mathcal{A}(K_j^m) \cap \mathbb{Z}^d$ we have $R_i < \frac{m}{6(\beta_d+1)}$.

Denote by $\|\cdot\|_\infty$ the maximum norm in \mathbb{Z}^d and in \mathbb{R}^d . The key observation is that the event “the level- m cube is passable” only depends on what happens in finitely many level- m cubes around it. More precisely:

Lemma 2.3. Suppose that $m > 6$ and $\|i - j\|_\infty \geq 5$. Then the events $\{K_i^m \text{ is passable}\}$ and $\{K_j^m \text{ is passable}\}$ are independent.

Proof of Lemma 2.3. Consider $\ell_1 \in \mathcal{A}(K_i^m) \cap \mathbb{Z}^d$ and $\ell_2 \in \mathcal{A}(K_j^m) \cap \mathbb{Z}^d$. By Lemma 2.2, the event $\{R_{\ell_k} < \frac{m}{6(\beta_d+1)}\}$ only depends on what happens inside $B_{\ell_k}(\frac{\beta_d m}{6(\beta_d+1)})$, $k = 1, 2$. Note that $B_{\ell_k}(\frac{\beta_d m}{6(\beta_d+1)}) \subset B_{\ell_k}(m/6)$. It is then straightforward to check that, if $m > 6$ and $\|i - j\|_\infty \geq 5$, for all such ℓ_1, ℓ_2 it holds that $B_{\ell_1}(m/6) \cap B_{\ell_2}(m/6) = \emptyset$, which concludes the proof of Lemma 2.3. \square

Denote $p_m := \mathbf{P}[K_0^m \text{ is passable}]$. Next, our goal is to show that if λ is small enough, then $p_m \rightarrow 0$ as $m \rightarrow \infty$.

Consider the event

$$A_n = \left\{ \text{in } \mathcal{A}(K_0^n) \text{ there exists a connected component of diameter at least } \frac{n}{2} \text{ of passable level-}(3 \log n) \text{ cubes} \right\}.$$

Lemma 2.4. We have, for $n > 6$,

$$\mathbf{P}[A_n] \leq \left(\frac{n}{\log n} \right)^d (11^d p_{3 \log n})^{k_0}, \quad (2.9)$$

where

$$k_0 = \left\lfloor \frac{n}{30\sqrt{d} \log n} \right\rfloor - 1.$$

Proof of Lemma 2.4. Since the diameter of a level- m cube is $m\sqrt{d}$, on the event A_n , there exist $m' \in \mathbb{Z}_+$, $i_1, \dots, i_{m'} \in \mathbb{Z}^d$ such that

- $K_{i_j}^{3 \log n} \subset \mathcal{A}(K_0^n)$ for all $j = 1, \dots, m'$,

- $\|i_j - i_{j-1}\|_\infty = 1$, for all $j = 2, \dots, m'$,
- and $\|i_0 - i_{m'}\|_\infty \geq \frac{n}{6\sqrt{d}\log n}$.

Then, define $\tau(1) := 1$, and

$$\tau(j) = \max\{\ell > \tau(j-1) : \|i_\ell - i_{\tau(j-1)}\|_\infty = 5\}$$

for $j = 2, \dots, k_0$ (indeed, since $5k_0 < \frac{n}{6\sqrt{d}\log n}$, we have that $\tau(k_0) \leq m'$). Then, the collection of level- $(3\log n)$ cubes $\gamma = (K_{i_{\tau(1)}}^{3\log n}, \dots, K_{i_{\tau(k_0)}}^{3\log n})$ has the following properties: $\gamma \subset \mathcal{A}(K_0^n)$, for $j = 1, \dots, k_0$ the cubes $K_{i_{\tau(j)}}^{3\log n}$ are passable, $\|i_{\tau(j)} - i_{\tau(j-1)}\|_\infty = 5$ and $\|i_{\tau(j')} - i_{\tau(j)}\|_\infty \geq 5$, for all $j \neq j'$. Intuitively, this collection corresponds to a “path” by passable cubes inside $\mathcal{A}(K_0^n)$; however, neighbouring elements of this path are not really neighbours, but they are separated enough to make them independent. The number of collections with such properties is at most $(\frac{n}{\log n})^d 11^{dk_0}$ (there are at most $(\frac{n}{\log n})^d$ possibilities for choosing the first cube in the collection, and then at each step there are at most $11^d - 9^d < 11^d$ possibilities for choosing the next one). For a fixed γ , by Lemma 2.3, the probability that all the cubes $K_{i_j}^{3\log n}$ in the collection γ are passable is at most $p_{3\log n}^{k_0}$, and so (2.9) holds. \square

Lemma 2.5. Suppose that, for some $n > e^{4\sqrt{d}}$, the cube K_0^n is passable and the following event occurs:

$$\left\{ \text{for all } i \in \mathcal{A}(K_0^n) \cap \mathbb{Z}^d \text{ it holds that } R_i < \frac{\log n}{2(\beta_d + 1)} \right\}. \quad (2.10)$$

Then, any level- $(3\log n)$ cube in $\mathcal{A}(K_0^n)$ intersecting with a connected component of $\mathcal{C}|_{\mathcal{A}(K_0^n)}$ with a diameter at least $n/2$ of painted level-1 cubes (cf. Definition 2.1(i)), and such that the distance from it to K_0^n is at most $n/2$, is passable, and, in particular, the event A_n occurs.

Proof of Lemma 2.5. Consider any level- $(3\log n)$ cube with the above properties, say $K_j^{3\log n}$. As $\rho(K_0^n, K_j^{3\log n}) \leq n/2$, we have $\mathcal{A}(K_j^{3\log n}) \subset \mathcal{A}(K_0^n)$ and thus for all $x \in \mathcal{A}(K_j^{3\log n})$ it holds that $R_x < \frac{\log n}{2(\beta_d + 1)}$. That is, the second condition in Definition 2.1 is satisfied. Let

$$\mathfrak{R}(K_j^{3\log n}) = \left\{ x \in \mathbb{R}^d : \inf_{y \in K_j^{3\log n}} \|x - y\|_\infty \leq 2\log n \right\}.$$

Use the abbreviation $r_1 := \frac{\log n}{2(\beta_d + 1)}$, and consider some level-1 cube $K_l^1 \subset \mathfrak{R}(K_j^{3\log n})$ such that $K_l^1 \subset \mathcal{C}|_{\mathcal{A}(K_0^n)}$. On the event (2.10) this means that there exists $i \in \mathcal{A}(K_0^n) \cap \mathbb{Z}^d$ such that $K_l^1 \subset B_i(r_1)$. By Lemma 2.2, the event $\{R_i \leq r_1\}$ only depends on the configuration inside $B_i(\beta_d r_1)$. Since $r_1 + \beta_d r_1 + 2\sqrt{d} = \frac{\log n}{2} + 2\sqrt{d} < \log n$ (we supposed that $\log n > 4\sqrt{d}$), we obtain that $B_i(\beta_d r_1)$ is fully inside $\mathcal{A}(K_j^{3\log n})$. So, $K_l^1 \subset B_i(R_i^{\mathcal{A}(K_j^{3\log n})})$, and, consequently, $\mathfrak{R}(K_j^{3\log n})$ intersects with a connected component of diameter at least $2\log n$ of level-1 cubes from $\mathcal{C}|_{\mathcal{A}(K_j^{3\log n})}$. This implies that $K_j^{3\log n}$ is passable. \square

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Using first Lemma 2.5, and then (2.8) together with Lemma 2.4, we obtain that

$$\begin{aligned} p_n &= \mathbf{P}[K_0^n \text{ is passable}] \\ &\leq \mathbf{P}[A_n] + \mathbf{P}\left[\text{there exists } i \in \mathcal{A}(K^n) \cap \mathbb{Z}^d : R_i \geq \frac{\log n}{2(\beta_d + 1)}\right] \\ &\leq \left(\frac{n}{\log n}\right)^d (11^d p_{3 \log n})^{k_0} + (3n)^d e^{-c'(\lambda) \log^d n}, \end{aligned} \quad (2.11)$$

where $c'(\lambda) = 2^{-d}(\beta_d + 1)^{-d}c(\lambda)$. Use the abbreviation $\varepsilon_d = 11^{-d}/2$. Choose a large enough m_0 such that

$$\begin{aligned} m &> e^{4\sqrt{d}}, \\ m^{1/2} &\leq \left\lfloor \frac{m}{30\sqrt{d} \log m} \right\rfloor - 1, \\ \frac{m}{\log m} &\leq 3m, \\ (11^d \varepsilon_d)^{m^{1/2}} &\leq e^{-\log^d m}, \\ \varepsilon_d &> (3m)^d \left((11^d \varepsilon_d)^{m^{1/2}} + e^{-\log^d m} \right) \end{aligned}$$

for all $m \geq m_0$ (note that in fact $11^d \varepsilon_d = 1/2$ and that $e^{4\sqrt{d}} > 6$). Choose a small enough λ in such a way that $c'(\lambda) \geq 1$ and also that $e^{-\lambda(3m_0)^d} > 1 - \varepsilon_d$. Note that the last condition on λ implies that for any $m \leq m_0$ we have $p_m < \varepsilon_d$ (this is because, with probability at least $e^{-\lambda(3m_0)^d}$, there will be no centers in $\mathcal{A}(K^m)$, in which case K^m is not passable).

Then, if $n > m_0$ and $p_{3 \log n} < \varepsilon_d$ we have by (2.11)

$$\begin{aligned} p_n &\leq \left(\frac{n}{\log n}\right)^d (11^d p_{3 \log n})^{k_0} + (3n)^d e^{-\log^d n} \\ &\leq \left(\frac{n}{\log n}\right)^d (11^d \varepsilon_d)^{k_0} + (3n)^d e^{-\log^d n} \\ &\leq (3n)^d (11^d \varepsilon_d)^{n^{1/2}} + (3n)^d e^{-\log^d n} \\ &< \varepsilon_d. \end{aligned}$$

By induction, this implies that $p_n < \varepsilon_d$ for all n (i.e., using the above calculation, first we obtain that $p_m < \varepsilon_d$ for all $m \leq m_0$ implies that $p_m < \varepsilon_d$ for all $m \leq e^{m_0/3}$, and so on). Moreover, using (2.11) once again, we obtain

$$\begin{aligned} p_n &\leq (3n)^d (11^d \varepsilon_d)^{n/2} + (3n)^d e^{-\log^d n} \\ &\leq 2(3n)^d e^{-\log^d n}, \end{aligned} \quad (2.12)$$

so $p_n \rightarrow 0$ as $n \rightarrow \infty$. Using (2.8), one can write (recall that $c(\lambda) > c'(\lambda) \geq 1$)

$$\begin{aligned} &\mathbf{P}[K_0^n \text{ intersects with a connected component of diameter at least } n/2 \\ &\quad \text{of painted level-1 cubes}] \\ &\leq \mathbf{P}[K_0^n \text{ is passable}] \end{aligned}$$

$$\begin{aligned}
& + \mathbf{P} \left[\text{there exists } i \in \mathcal{A}(K_0^n) \cap \mathbb{Z}^d \text{ such that } R_i \geq \frac{n}{6(\beta_d + 1)} \right] \\
& + \mathbf{P}[\text{there exists } i \in \mathbb{Z}^d \setminus \mathcal{A}(K_0^n) \text{ such that } R_i \geq \rho(\{i\}, K_0^n)] \\
& < p_n + (3n)^d e^{-(n/6(\beta_d+1))^d} + c_2 \sum_{\ell=n}^{\infty} \ell^{d-1} \mathbf{P}[R_\ell \geq \ell] \\
& \leq p_n + (3n)^d e^{-(n/6(\beta_d+1))^d} + c_2 \sum_{\ell=n}^{\infty} \ell^{d-1} e^{-\ell^d} \\
& \leq 2(3n)^d e^{-\log^d n} + (3n)^d e^{-(n/6(\beta_d+1))^d} + c_2 \sum_{\ell=n}^{\infty} \ell^{d-1} e^{-\ell^d} \\
& \rightarrow 0,
\end{aligned} \tag{2.13}$$

as $n \rightarrow \infty$.

We proved that we can choose λ small enough to obtain

$$\mathbf{P}[K_0^n \text{ intersects with a connected component of diameter at least } n/2 \text{ of painted level-1 cubes}] \rightarrow 0,$$

as $n \rightarrow \infty$. Note that, since $\mathcal{C} \subset \mathfrak{C}$,

$$\begin{aligned}
& \mathbf{P}[K_0^n \text{ intersects with a connected component of diameter at least } n/2 \\
& \text{ of painted level-1 cubes}] \\
& \geq \mathbf{P}[0 \text{ belongs to an unbounded connected subset of } \mathcal{C}]
\end{aligned}$$

and the latter probability is strictly positive, in the case when there is percolation. So, there is no percolation for λ small enough and the part (i) of [Theorem 1.1](#) is proved.

As for the part (ii), we proceed as follows. Denote by $H_2 \subset \mathbb{R}^d$ the two-dimensional plane:

$$H_2 = \{x = (x^{(1)}, \dots, x^{(d)}) \in \mathbb{R}^d : x^{(3)} = \dots = x^{(d)} = 0\}.$$

Now one can write

$$\begin{aligned}
& \{\text{there is no unbounded connected subset in } \mathbb{R}^d \setminus \mathcal{C}\} \\
& \subset \{\text{for any bounded } W \subset H_2, \text{ there is a contour around } W \text{ in } \mathcal{C} \cap H_2\} \\
& \subset \{\text{for any bounded } W \subset H_2, \text{ there is a contour around } W \text{ in } \mathfrak{C} \cap H_2\} \\
& \subset \{\text{for an infinite number of cubes } K_0^n, n = 1, 2, 3, \dots, K_0^n \text{ intersects} \\
& \text{ with a connected component of diameter at least } n/2 \\
& \text{ of painted level-1 cubes}\}.
\end{aligned}$$

By (2.13) and Borel–Cantelli lemma, for small enough λ the probability of the last event is 0, and thus part (ii) of [Theorem 1.1](#) is proved. \square

Acknowledgements

S.P. and M.V. are grateful to CNPq (302981/02-0 and 306029/03-0) for partial support. M.V.F. acknowledges the support by CNPq (150989/05-9) and FAPESP (2005/00248-6). This research was developed using resources of CENAPAD-SP (National Center of High Performance Computing at São Paulo), project UNICAMP/FINEP-MCT, Brazil. The authors are grateful to

Yuval Peres, who suggested this problem to them, and to Daniel Andrés Díaz Pachón, who read the first version of the manuscript very carefully and pointed out several mistakes. Also, the authors thank the anonymous referee for valuable comments and suggestions.

References

- [1] C. Hoffman, A.E. Holroyd, Y. Peres, A stable marriage of Poisson and Lebesgue, *Ann. Probab.* 34 (4) (2006).
- [2] C. Hoffman, A.E. Holroyd, Y. Peres, Tail bounds for the stable marriage of Poisson and Lebesgue, 2005. Available at arXiv.org as [math.PR/0507324](https://arxiv.org/abs/math.PR/0507324).
- [3] M.V. Menshikov, S.Yu. Popov, M. Vachkovskaia, On the connectivity properties of the complementary set in fractal percolation models, *Probab. Theory Related Fields* 119 (2001) 176–186.
- [4] M.D. Penrose, Continuum percolation and Euclidean minimal spanning trees in high dimensions, *Ann. Appl. Probab.* 6 (2) (1996) 528–544.